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SOLIDIFICATION OF A LIQUID ABOUT A CYLINDRICAL PIPE

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Abstract – The temperature distribution and the rate of removal of heat by a coolant are predicted for the process of solidification of a liquid about a cold, isothermal pipe. The heat balance integral method incorporating spacial sub-division is used. It is found that acceptable results can be obtained by using only a small number of sub-divisions together with a piece-wise, linear profile. Furthermore, the results illustrate that the sensitivity which is normally associated with the heat balance integral method is overcome.

NOMENCLATURE

- a, external radius of pipe;
- c, heat capacity;
- F, flux removed by coolant;
- K, conductivity;
- L, latent heat of freezing;
- n, number of sub-divisions;
- r, radial coordinate;
- R, position of the solid–liquid interface;
- t, time;
- T, temperature in solid;
- T_0, freezing temperature of liquid;
- T_s, surface temperature of pipe;
- u, non-dimensional temperature;
- u_i, ith isotherm;
- z, r/a;
- Z, R/a;
- Z_i, position of ith isotherm.

Greek symbols

- α, latent heat parameter $= L/c(T_0 - T_s)$;
- κ, thermal diffusivity;
- λ_j, jth coefficient in small time expansion of Z;
- μ_{ij}, jth coefficient in small time expansion of Z_i;
- ρ, density;
- τ, dimensionless time $= \kappa t/a^2$.

INTRODUCTION

IN A RECENT paper [1] the author proposed a refinement of the heat balance integral method suitable for the solution of one-dimensional, phase change problems. The technique described in [1] attempts to maintain the flexibility and simplicity of the original method [2] and yet provide scope for improved precision. The effectiveness of such a refinement was illustrated for the classical 'melting of ice' problem for which there exists an analytic solution. This model problem afforded a major simplification to the method in that a similarity variable could be introduced and so reduce the approximating differential system into a simpler, algebraic one. In the present paper the performance of the technique, when applied to a problem for which no such similarity variable can be found, is discussed.

The problem to be considered is the radial growth of a solid phase about a cylindrical pipe, whose surface temperature is maintained at a value below the freezing temperature of the surrounding liquid. A specification of the problem appears in the next section.

There are many suitable numerical techniques already available to the engineer. However, as recognised by Churchill and Gupta [3], there is still a need for simple and efficient approximate methods. In fact Churchill and Gupta propose such a method which, from independent applications of their procedure, provides an estimate of both the incident heat flux and the position of the solid–liquid interface. The method proposed in [1] supplies all this information automatically. Hence, it is the intention of this investigation to demonstrate the power of the heat balance integral approach when spacial sub-division is incorporated. It will be found that for most practical purposes the number of sub-divisions required is small.

Before pursuing the details, it is worth commenting briefly on the choice of temperature profile. It is well known that the original method described by Goodman [2] is sensitive to variations in the shape of the approximating temperature profile. Sparrow [4] reports the inadequacy of polynomial profiles for cylindrical problems. For such situations logarithmic profiles are recommended by Lardner and Pohle [5], although the application of anything other than the simplest of these is extremely tedious. For the sake of simplicity, a piece-wise, linear profile will be employed in the following investigation. The incorporation of spacial sub-division overcomes the sensitivity previously observed in the heat balance integral method and makes judicious profile selection unnecessary.

PROBLEM SPECIFICATION

Consider the idealised problem of the solidification of a liquid, initially at its freezing temperature T_0,

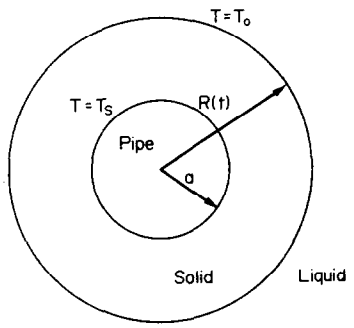

FIG. 1. Problem specification.

about a cylindrical pipe of radius a, maintained at a constant, lower temperature T_s by coolant flowing through the pipe. It is assumed that the physical properties of the material remain constant throughout the process and there is no change of volume on solidification. The process may be described as

$$\frac{\partial T}{\partial t} = \kappa \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \quad a < r < R(t), \quad t > 0;$$

$$T = T_s, \quad r = a, \quad t \geq 0;$$

$$T = T_0, \quad r \geq R(t), \quad t > 0;$$

and

$$K\left(\frac{\partial T}{\partial r}\right)_R = L\rho\frac{dR}{dt},$$

where T is temperature and R is the position of the solid–liquid interface. The constants κ, K, ρ and L are thermal diffusivity, conductivity, density and latent heat of freezing, respectively. It is further assumed that the pipe is initially cooled sufficiently rapidly that a discontinuous change in the temperature takes place at $r = a$, when $t = 0$. This provides the initial condition

$$R(0) = a$$

which completes the specification of the problem.

It is convenient to non-dimensionalise the above set of equations by introducing the variables

$$u = (T - T_s)/(T_0 - T_s),$$

$$z = r/a,$$

$$\tau = \kappa t/a^2,$$

and

$$Z(\tau) = R(t)/a.$$

The above equations become

$$\frac{\partial u}{\partial \tau} = \frac{1}{z}\frac{\partial}{\partial z}\left(z\frac{\partial u}{\partial z}\right), \quad 1 < z < Z(\tau), \quad \tau > 0; \tag{1}$$

$$u = 0, \quad z = 1, \quad \tau \geq 0; \tag{2}$$

$$u = 1, \quad z \geq Z(\tau), \quad \tau > 0; \tag{3}$$

$$\left(\frac{\partial u}{\partial z}\right)_Z = \alpha\frac{dZ}{d\tau}; \tag{4}$$

and

$$Z(0) = 1, \tag{5}$$

where α is a dimensionless latent heat parameter defined by

$$\alpha = \frac{L}{c(T_0 - T_s)}.$$

As indicated in [1] a starting solution is necessary in order to implement the heat balance technique. The appropriate small time solution is given by Poots [6] and the initial motion of the front is of the form

$$Z(\tau) = 1 + \lambda_0\tau^{1/2} + \lambda_1\tau + \lambda_2\tau^{3/2} + \ldots.$$

Expressions for the coefficients λ_0, λ_1 and λ_2 are given by Professor Poots although the expression for λ_2 contains a number of typographical errors.

THE HEAT BALANCE INTEGRAL METHOD

The technique described in [1] involves the sub-division of the temperature range into equal intervals. A penetration variable is associated with each isotherm created by the sub-division. Using the heat balance integral approach, a system of differential equations for the penetration variables is produced. Consequently, the solution of the system gives the motion of each isotherm, one of which corresponds to the solidification front.

The same technique is adopted here. Consider the non-dimensional form of the problem, equations (1) to (5), and sub-divide the temperature range into n equal intervals so that

$$u_i = i/n, \quad i = 0, 1, \ldots, n.$$

The position of each isotherm u_i is denoted by the penetration variable $Z_i(\tau)$, where $Z_n(\tau)$ is the depth of solidification and $Z_0(\tau) = 1$ for all τ. Equation (1) is multiplied throughout by z and integrated over each sub-region $[Z_i, Z_{i+1}]$, $i = 0, 1, \ldots, n-1$, in turn, producing n heat balance equations. The normalised temperature u is approximated by the linear profile

$$\frac{i}{n} + \frac{(z - Z_i)}{n(Z_{i+1} - Z_i)} \tag{6}$$

in the region $[Z_i, Z_{i+1}]$, $i = 0, 1, \ldots, n-1$. Replacing u by the appropriate profile in each heat balance equation and ensuring that expressions representing change in flux are approximated by the discontinuous change in adjacent profile gradients, the following system of equations is obtained:

$$(2Z_1 + 1)\dot{Z}_1 = \frac{6}{Z_1 - 1} - \frac{6Z_1}{Z_2 - Z_1},$$

$$(2Z_{i+1} + Z_i)\dot{Z}_{i+1} + (2Z_i + Z_{i+1})\dot{Z}_i = \frac{6Z_i}{Z_{i+1} - Z_i} - \frac{6Z_{i+1}}{Z_{i+2} - Z_{i+1}}, \quad i = 1, \ldots, n-2, \tag{7}$$

and

$$[2(1 + 3n\alpha)Z_n + Z_{n-1}]\dot{Z}_n + (Z_n + 2Z_{n-1})\dot{Z}_{n-1} = \frac{6Z_{n-1}}{Z_n - Z_{n-1}}.$$

Given the values of Z_i for some suitably small value of τ, the above system (7) can be solved fairly easily using a standard Runge–Kutta procedure.

The starting values are obtained by further consideration of the solution obtained by Poots [6]. The initial motion of each isotherm is assumed to have a form similar to that of the solidification front, namely,

$$Z_i(\tau) = 1 + \mu_{i0}\tau^{1/2} + \mu_{i1}\tau + \mu_{i2}\tau^{3/2} + \ldots . \quad (8)$$

Substituting the expansion for $Z_i(\tau)$ into Poots' solution and expanding in terms of small τ produces expressions for the coefficients μ_{ij}. The process is extremely laborious and an alternative starting procedure is realised in the following discussion.

NUMERICAL RESULTS

The system of equations (7), with latent heat parameter $\alpha = 1$, was solved numerically using a standard Runge–Kutta procedure. A suitable size of time step was selected by requiring the solution obtained to agree to four decimal places with that produced with a time step of half the size. The number of sub-divisions, n, was also varied and the estimates obtained for the depth of solidification at two particular stages are presented in Table 1. Also included is an estimate based on a single profile of the form

$$\log z/\log Z,$$

as suggested in [5]. For the purposes of a meaningful comparison, results obtained by the Isotherm Migration Method (see Crank [7]) are quoted. The Isotherm Migration Method was applied with a very small time step and a slightly modified finite-difference approximation at $u = 1$ in an attempt to produce a reliable estimate of the depth of solidification.

Each method was started from an initial condition, obtained from the series expansion discussed in the previous section, corresponding to a time $\tau = 0.01$.

The results presented illustrate once again the effectiveness of the heat balance integral approach when sub-division is incorporated. It is interesting to note that the estimates obtained using just two sub-regions and linear profiles are comparable with the results produced from a single logarithmic profile, as suggested by Lardner and Pohle.

The disadvantage of the procedure as presented here is that as the number of sub-divisions, n, is increased the size of the acceptable time increment, for the early stages of computation, decreases rapidly. A time increment of 0.0002 was found to be suitable for the case $n = 4$ and was maintained in order to preserve consistency of accuracy. For larger values of n it is more efficient to use a much simpler differential equation solver instead of a Runge–Kutta package. However, for the case $n = 4$ the computing time required is not unreasonable. Also, the accuracy produced using just four sub-regions is sufficient for most practical situations. The motion of the solidification front, predicted by the above procedure, is illustrated in Fig. 2. In Fig. 3, the flux, F, emerging from the solid is plotted against $\sqrt{(\kappa t)}/a$. An estimate of the flux can easily be obtained from the solution of the system of equations (7), since

$$F = K\left(\frac{\partial T}{\partial r}\right)_{r=a} = \frac{K(T_0 - T_s)}{a}\left(\frac{\partial u}{\partial z}\right)_{z=1} \simeq \frac{K(T_0 - T_s}{an(Z_1 - 1)}$$

by virtue of expression (6). Included in Fig. 3 are estimates produced using the technique proposed by Churchill and Gupta [3]. The two approximate solutions are observed to be in close agreement. Unfortunately, such a comparison cannot be made in Fig. 2 because, as Churchill and Gupta point out, $R(t)$ cannot be obtained by their technique when the whole of the liquid phase is at the solidification temperature.

Finally, it is interesting to examine the small time

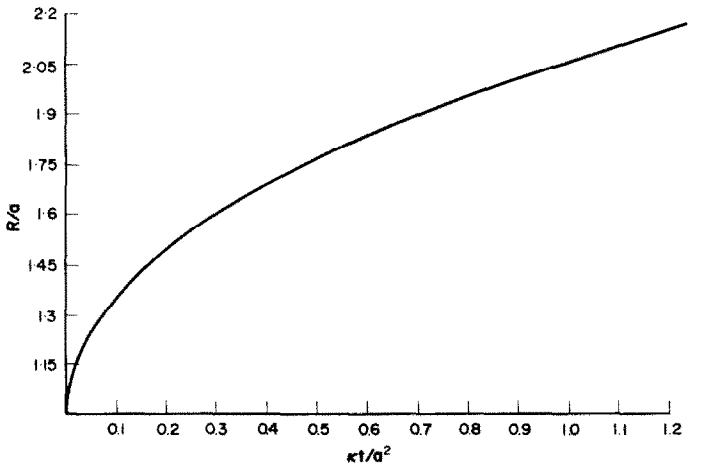

FIG. 2. Depth of solidification.

Table 1.

Time $\kappa t/a^2$	Method	Depth of solidification $R(t)/a$
0.05	Equations (7) with $n = 1$	1.2460
	$n = 2$	1.2548
	$n = 4$	1.2610
	$n = 8$	1.2638
	I.M.M.	1.2695
	Lardner and Pohle [5]	1.2526
0.1	Equations (7) with $n = 1$	1.3367
	$n = 2$	1.3526
	$n = 4$	1.3629
	$n = 8$	1.3672
	I.M.M.	1.3769
	Lardner and Pohle [5]	1.3543

$\alpha = 1$.

behaviour of the approximating system (7) and compare it with the known analytic behaviour. If the solution of (7) is assumed to have a small time expansion of the form given in (8) then estimates of the coefficients μ_{ij} can be obtained for different values of n. The first two coefficients in each expansion were calculated for the cases $n = 1$, 2 and 4. The values corresponding to two of the isotherms are presented in Table 2.

It is encouraging to observe that the discrepancy between each estimate and the corresponding exact value is small. Also, as the number of sub-divisions increases the errors decrease, albeit slowly. These results suggest the possibility of using the small time behaviour of the approximate system to start the numerical solution in situations where the exact behaviour is either unknown or difficult to determine. This is illustrated in the next section where a related problem is briefly considered.

Table 2.

Isotherm: Estimate of:	$Z_{\frac{1}{2}n}$ $\mu_{\frac{1}{2}n0}$	$\mu_{\frac{1}{2}n1}$	Z_n μ_{n0}	μ_{n1}
$n = 1$	—	—	1.1547	0.3951
$n = 2$	0.5671	0.2768	1.1838	0.2866
$n = 4$	0.5648	0.2506	1.2088	0.2451
Exact values	0.5635	0.2358	1.2401	0.2040

THE EXTERNAL CYLINDER PROBLEM

Suppose in the problem specified earlier, the latent heat is taken to be zero (i.e. $\alpha = 0$). The system of equations (7) now represents an approximation to the process of conduction in a region external to a cylinder. The region is initially at constant temperature T_0 and the surface of the cylindrical pipe is maintained at a lower, constant temperature T_s. An exact solution to this problem is given by Carslaw and Jaeger [8] and has been tabulated by Jaeger [9]. This solution is presented in Fig. 4.

The system of equations (7) was solved numerically with $\alpha = 0$ and $n = 4$. Starting values were obtained by the procedure suggested in the previous section. The initial behaviour obtained for Z_n is somewhat meaningless since the isotherm T_0 is theoretically at infinity for all time $\tau > 0$. However, the results given in Fig. 4 are impressive at all points except those in the locality of Z_n. Clearly, the estimate of the temperature at such points is significantly influenced by the finite speed of penetration that is predicted for Z_n, for all τ. The larger the radii considered the greater is the discrepancy observed and this is to be expected in a situation where the concept of penetration depth is perhaps inappropriate. Nevertheless, the results obtained compare favourably with the exact solution as tabulated by Jaeger.

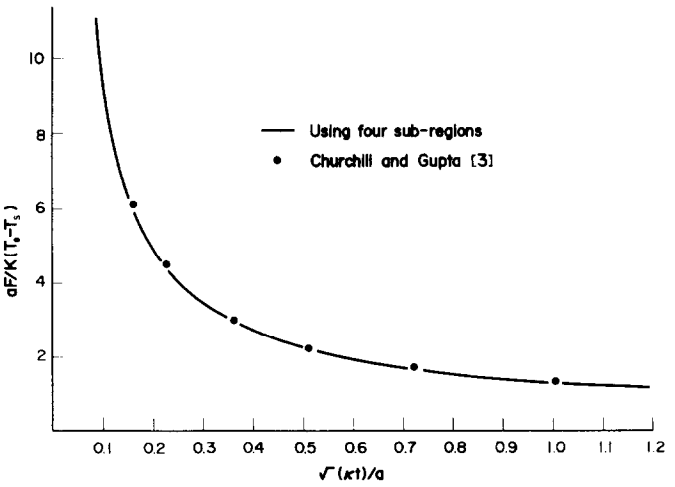

FIG. 3. Rate of removal of heat per unit area by coolant.

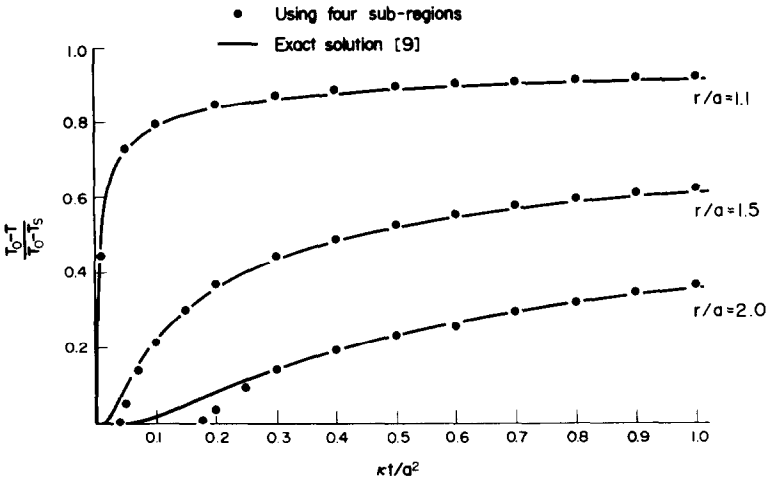

FIG. 4. External cylinder problem – variation in temperature with time at specified radii.

CONCLUDING REMARKS

A refinement of the heat balance integral method, as described by the author in [1], has been successfully applied to the problem of the solidification of a liquid about a cylindrical pipe. Although a number of assumptions have been made in the problem specification, the results obtained are relevant to many industrial problems. It has been shown that acceptable estimates of both the temperature and the flux can be obtained by using a small number of sub-divisions and linear profiles. In addition, the approximating differential system that arises can be solved by well established numerical procedures. It has also been observed that the technique can be considered as self-starting for situations where small time expansions are not available.

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SOLIDIFICATION D'UN LIQUIDE AUTOUR D'UN TUBE CYLINDRIQUE

Résumé—La distribution de température et le flux thermique sont calculés dans le mécanisme de solidification d'un liquide autour d'un tube froid et isotherme. On utilise la méthode intégrale du bilan thermique qui introduit une subdivision spatiale. On trouve que des résultats acceptables peuvent être obtenus en utilisant seulement un petit nombre de subdivisions avec un profil linéaire par morceaux. Les résultats montrent que la sensibilité qui est normalement associée à la méthode intégrale est maitrisée.

ERSTARREN EINER FLÜSSIGKEIT AN EINEM ZYLINDRISCHEN ROHR

Zusammenfassung—Die Temperaturverteilung und die durch ein Kühlmittel abgeführte Wärmemenge werden für den Erstarrungsprozeß einer Flüssigkeit mit Hilfe der Wärmebilanzintegral-Methode mit räumlicher Unterteilung bestimmt. Befriedigende Ergebnisse erhält man schon bei einer kleinen Anzahl von Raum-Elementen und einem stückweise linearen Temperaturprofil.

Ferner zeigen die Ergebnisse, daß die Empfindlichkeit, die normalerweise bei der Wärmebilanzintegral-Methode auftritt, überwunden werden konnte.

ЗАТВЕРДЕВАНИЕ ЖИДКОСТИ ВОКРУГ ЦИЛИНДРИЧЕСКОЙ ТРУБКИ

Аннотация — Проведен расчет распределения температуры и скорости отвода тепла теплоносителем для процесса затвердевания жидкости вокруг холодной изотермической трубки. Использован интегральный метод теплового баланса, включающий разбиение объема на подобласти. Найдено, что с помощью только небольшого числа разбиений и кусочнолинейного профиля можно получить приемлемые результаты. Показано также, что таким образом можно избежать неустойчивости, обычно связанной с использованием интегрального метода теплового баланса.